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# COMPARISON OF CEP ESTIMATORS FOR ELLIPTICAL NORMAL ERRORS

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## Introduction

A common parameter for describing the accuracy of a weapon is the circular probable error, generally referred to as CEP. CEP is simply the bivariate analog of the probable error of a single variable and measures the radius of a mean centered circle which includes 50% of the bivariate probability mass. In the case of circular normal errors where the error variances are the same in both directions, CEP can be expressed in terms of the common standard deviation, and estimators are easily formulated and compared. In the case of elliptical normal errors, CEP cannot be expressed in closed form, and hence, estimators are less easily formulated. The problem addressed herein is the comparison of CEP estimators for the elliptical case based on some of the commonly used CEP approximations.

It will be instructive to first review the case of circular normal errors. In general, it will be assumed that the errors in the X and Y directions are independent with mean zero and variances  $\sigma_x^2$  and  $\sigma_y^2$ , respectively. Under the circular normal assumption,  $\sigma_x^2 = \sigma_y^2 = \sigma^2$  and the bivariate distribution of errors is given by

$$f_c(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}, \quad -\infty < x,y < \infty. \quad (1)$$

The distribution of  $R = (X^2 + Y^2)^{1/2}$  is easily derived and found to be

$$g_c(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r > 0. \quad (2)$$

This is the well known Rayleigh distribution with cumulative distribution function

$$P(R < r) = G_c(r) = 1 - e^{-r^2/2\sigma^2}. \quad (3)$$

By definition,  $G_c(\text{CEP}) = .5$ , and the solution of equation (3) yields the well-known expression

$$\text{CEP} = [-2\ln(.50)]^{1/2} \sigma = 1.1774\sigma. \quad (4)$$

Four estimators for CEP in the circular case were formulated and compared by Moranda (1959).

Consider now the case of elliptical normal errors. Here the variances are unequal, and the bivariate distribution of errors is given by

$$f_E(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}[(x/\sigma_x)^2 + (y/\sigma_y)^2]}, \quad -\infty < x,y < \infty. \quad (5)$$

For this case, the distribution of the radial error  $R$  was derived by Chew and Boyce (1961) and has form

$$g_E(r) = \frac{r}{\sigma_x\sigma_y} e^{-ar^2} I_0(br^2) \quad (6)$$

where

$$a = \frac{\sigma_y^2 + \sigma_x^2}{(2\sigma_x\sigma_y)^2}, \quad b = \frac{\sigma_y^2 - \sigma_x^2}{(2\sigma_x\sigma_y)^2}$$

and  $I_0$  is the modified Bessel function of the first kind and zero order, i.e.,

$$I_0(x) = \frac{1}{\pi} \int_0^{\pi} e^{-x \cos \theta} d\theta.$$

The cumulative distribution function for R is denoted by

$$P(R < r) = G_E(r) = \int_0^r g_E(t) dt, \quad (7)$$

but it cannot be expressed in closed form. Hence, the radius of the 50% circle for the elliptical case cannot be expressed by a simple formula as it was in the circular case. One has to solve  $G_E(\text{CEP}) = .5$  by numerical methods or by referring to tables prepared by Harter (1960), DiDonato and Jarnagin (1962), and others. To avoid using these tables or numerical procedures for CEP evaluation, a number of approximations have been developed over the years. Five of these approximations have been chosen for examination. They are designated below as  $\text{CEP}_1$  through  $\text{CEP}_5$ :

$$\text{CEP}_1 = 1.1774 \left( \frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{1/2} \quad (8)$$

$$\text{CEP}_2 = 1.1774 \left( \frac{\sigma_x + \sigma_y}{2} \right) \quad (9)$$

$$\text{CEP}_3 = \left( 2 \chi_{v, .50}^2 / v \right)^{1/2} \left( \frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{1/2} \quad (10)$$

$$v = \frac{(\sigma_x^2 + \sigma_y^2)^2}{\sigma_x^4 + \sigma_y^4}$$

$$CEP_4 = .565 \sigma_{\max} + .612 \sigma_{\min}, \sigma_{\min}/\sigma_{\max} \geq .25 \quad (11)$$

$$= .667 \sigma_{\max} + .206 \sigma_{\min}, \sigma_{\min}/\sigma_{\max} < .25$$

$$CEP_5 = \left[ 2^{\frac{1}{3}} \left( 1 - \frac{2}{9V} \right) \right]^{\frac{3}{2}} \left( \frac{\sigma_x^2 + \sigma_y^2}{2} \right)^{\frac{1}{2}}. \quad (12)$$

$CEP_1$  and  $CEP_2$  were taken from Groves (1961),  $CEP_3$  was established by Grubbs (1964),  $CEP_4$  is a piece-wise linear combination of the standard deviations, and  $CEP_5$  was also established by Grubbs (1964) using a Wilson-Hilferty transformation of the chi-square in  $CEP_3$ . Plots of each approximation versus the true CEP as a function  $\sigma_{\min}/\sigma_{\max}$  are shown in Figures 1 through 5. These give a fairly good indication of how well each performs. It is seen that  $CEP_1$  deteriorates rapidly as we depart from the circular case (for which  $CEP_1$  degenerates to  $1.1774\sigma$ ),  $CEP_2$  is reasonably good if the ratio  $\sigma_{\min}/\sigma_{\max}$  is not less than about .2,  $CEP_3$  appears good for all ratios, and  $CEP_4$  and  $CEP_5$  appear good to a lesser extent for all ratios.

If these approximations were used only as approximations for assumed values of the error variances (as one does in wargaming and round requirement studies), then there would be no estimation problem. However, in many cases, weapons analysts are using estimates of the variances in these approximations (based on sample data) to form estimates of CEP. Hence, the problem now becomes an estimation problem instead of an approximation problem. In particular, the problem addressed in this paper is that of comparing the five estimators for CEP formed by replacing the population variances in equations (8) through (12) with sample variances  $S_x^2 = \sum X_i^2/n$  and  $S_y^2 = \sum Y_i^2/n$ . (In these expressions,  $X_i$  and  $Y_i$  are the recorded errors in the X and Y directions, respectively, for the  $i$ th impact and  $n$  is the number of sample impacts.) These estimators will be referred to as  $\hat{CEP}_1$  through  $\hat{CEP}_5$  in the discussion which follows.

### Methodology

Measures of comparison employed in this study were the mean squared error (MSE), expected confidence interval length, and confidence interval confidence. With regard to the former, the MSE of an estimator  $\hat{\theta}$  for a parameter  $\theta$  is defined in the usual sense, i.e.,

$$\text{MSE}(\hat{\theta}) = E(\hat{\theta} - \theta)^2 = V(\hat{\theta}) + B^2(\hat{\theta})$$

where  $E$  represents expectation,  $V$  represents variance and  $B$  represents bias. It was chosen because it accounts for bias as well as variance and all five estimators are biased for CEP except in the degenerate circular case. With regard to the second measure, it was chosen because it too accounts for bias as well as variance but in the sense of interval estimation vice point. These computations were based on approximate distributions of CEP estimators and did not provide intervals with specified confidence in all cases. Hence, the third measure was included to estimate the true confidence.

The computation of these measures was straightforward but not simple due to the complexity of the estimators. Recall that 1, 3 and 5 each involve radicals of linear combinations of sample variances and estimators 2 and 4 involve linear combinations of sample standard deviations. Hence, the sampling distributions were approximated. The approximations were achieved by matching the variance of estimators 3 and 5 with the variance of the chi-square distribution and by matching the variance of estimators 2 and 4 with the variance of the chi distribution. Estimator 1 was simply approximated by a chi-square with  $2n$  degrees of freedom. This distribution is exact only in the circular case and was included to show how poorly it becomes when eccentricity of the distribution increases. The approximations are shown in Figure 6 and are discussed in more detail in the next paragraph.

Figure 6 provides a summary of approximate distributions for each  $\hat{CEP}_i$  and defines several multiplicative factors,  $K_i$ , eccentricity  $c$  and degrees of freedom  $\nu$  and  $\nu'$ .  $\nu^*$  does not have a simple form and is described below.

Because estimators 3 and 5 are of the same general form, the distribution of the squares of both was approximated by matching the variance of

$$\frac{\nu' (S_x^2 + S_y^2)}{(\sigma_x^2 + \sigma_y^2)} \quad (13)$$

with  $2\nu'$ , the variance of a chi-square with  $\nu'$  degrees of freedom. It was found that  $\nu' = n\nu$  where  $\nu = \frac{(c^2 + 1)^2}{c^4 + 1}$ . Expression (13) can be rewritten as

$$\frac{\nu' \hat{CEP}_i^2}{\sigma_y^2 K_i \left( \frac{c^2 + 1}{2} \right)} \quad \text{where } i = 3 \text{ or } 5 \quad (14)$$

to conform to the expressions in Figure 6.

Estimators 2 and 4, representing linear combinations of the standard deviations, were approximated by matching the variance of a chi with  $\nu^*$  degrees of freedom with the variance of

$$\frac{(\nu^*)^{\frac{1}{2}} (S_x + S_y)}{\sigma_x + \sigma_y} \quad (15)$$

The variance of expression (15) was found to be

$$\nu^* (1 - H^2(n)) \frac{1+c^2}{(1+c)^2} \quad (16)$$

and the variance of a chi with  $\nu^*$  degrees of freedom is

$$\nu^* (1 - H^2(\nu^*))$$

$$\text{where } H(x) = \sqrt{\frac{2}{x}} \frac{\Gamma(\frac{x+1}{2})}{\Gamma(\frac{x}{2})}$$

Upon equating the two, we find that  $v^*$  satisfies

$$H(v^*) = \left\{ 1 - (1 - H^2(n)) \frac{1+c^2}{(1+c)^2} \right\}^{\frac{1}{2}}$$

and can be obtained from a table of inverse solutions of the H function.

The approximate distributions allow one to derive approximate mean squared errors for the  $\hat{CEP}_i$  estimators which are given in Figure 7. The  $K_i$  coefficients are defined as before and  $\text{Bias}(\hat{CEP}_i)$  is defined as

$$E(\hat{CEP}_i) - \text{True } CEP_i \text{ for } i = 1, 2, \dots, 5.$$

In general,  $\sigma_y$  is a scale factor representing the maximum  $\sigma$  value; however, in the examples given here  $\sigma_y$  is always equal to 1. Note that  $\text{MSE}(\hat{CEP}_2)$  and  $\text{MSE}(\hat{CEP}_4)$  can be expressed in exact rather than approximate form.

Since a point estimate may not provide adequate information, approximate 95% confidence intervals were constructed for each estimator using the distributions discussed above. The approximate  $100(1-\alpha)\%$  confidence limits for CEP are given by

$$\left[ \frac{\hat{CEP}_i}{\left( \chi_{v_i, 1-\alpha/2}^2 / v_i \right)^{\frac{1}{2}}}, \frac{\hat{CEP}_i}{\left( \chi_{v_i, \alpha/2}^2 / v_i \right)^{\frac{1}{2}}} \right]$$

where  $\hat{CEP}_i$  is the  $i$ th estimator and  $v_i$  equals the degrees of freedom associated with  $\hat{CEP}_i$ . Expected confidence interval widths can then be computed and used as measures of comparison between estimators. Clearly, if one could compute exact 95% confidence intervals, comparison of interval widths would be straightforward. However, only approximate intervals can be obtained and the confidence associated with each interval must be computed before a complete evaluation can be made.

Confidence was estimated using 10,000 Monte Carlo replicates for samples of size 5, 10 and 20 and measuring the percentage of time the true CEP fell within the interval. Confidence and expected confidence interval widths were then jointly examined.

### Results

The object of this study was to examine and evaluate the behavior of several candidate CEP estimators over a wide range of conditions. Sample sizes ranged from 5 to 400 and eccentricities ranged from  $c = 1$ , the circular case to  $c = 20$ , a highly elliptical case. Extreme values of the sample size and eccentricity may be infrequently encountered but were included for completeness. Clearly, an estimator behaving poorly under circumstances unlikely to be observed should not be disregarded as a viable candidate.

Prior to determining approximate distributions and mean squared error (MSE) approximations for the estimators, a Monte Carlo simulation was developed for computing the variance, bias, average squared error (ASE) and standard error for each estimator at each of three sample sizes ( $n = 5, 10, 20$ ). The simulated ASE's were used as a check against MSE approximations which were subsequently computed.

Upon comparing the simulated ASE's against results of the MSE approximations for sample sizes 5, 10, and 20, it became evident that MSE approximations were inadequate for estimators 3 and 5. In fact, in the mid-range of the eccentricity,  $c$ , the MSE for 3 and 5 differed from the simulated values of ASE by as much as three times the standard error. For this reason, the simulated ASE values are presented in Figure 8 while the approximate MSE values, found suitable for larger sample sizes, are shown in Figure 9.

Despite some fluctuation at  $c = .05$ , Figures 8 and 9 show estimators 2 through 5 producing fairly close results. As the sample size increased,

estimator 4 exhibited the smallest mean squared error and appeared to be the most satisfactory point estimator.

Figures 10 and 11 contain expected confidence interval width and confidence interval confidence, respectively. If the computation of exact 95% confidence intervals were possible, a straightforward selection of the estimator producing the narrowest width could be made. However, the approximate confidence intervals have varying levels of confidence associated with them, all of which underestimate or overestimate the desired 95% level. It appears that the wider lengths are associated with higher confidence and the narrower widths with the lower confidences so that a true comparison is not really possible. However, it is evident that estimators 2 through 5 do not distinguish themselves as being far superior or grossly inferior to one another. This is essentially the same result obtained from the MSE comparisons.

In summary, unless  $c$  is very small, estimators 2 through 5 produce reasonably close results. If confidence intervals are not desired, estimator 4 would be an acceptable choice. Otherwise, estimator 3 is recommended due to ease of confidence interval computability.

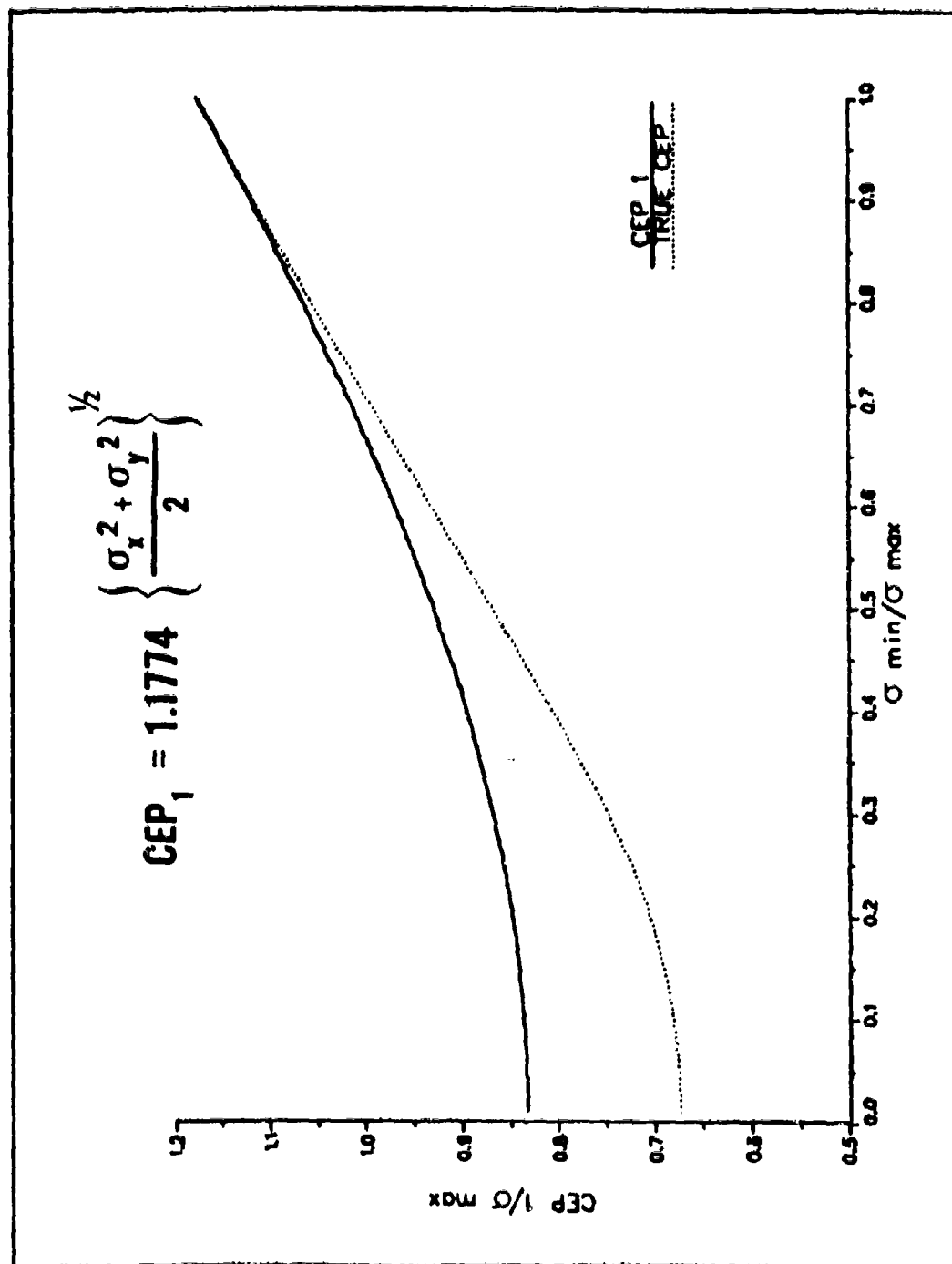


Figure 1

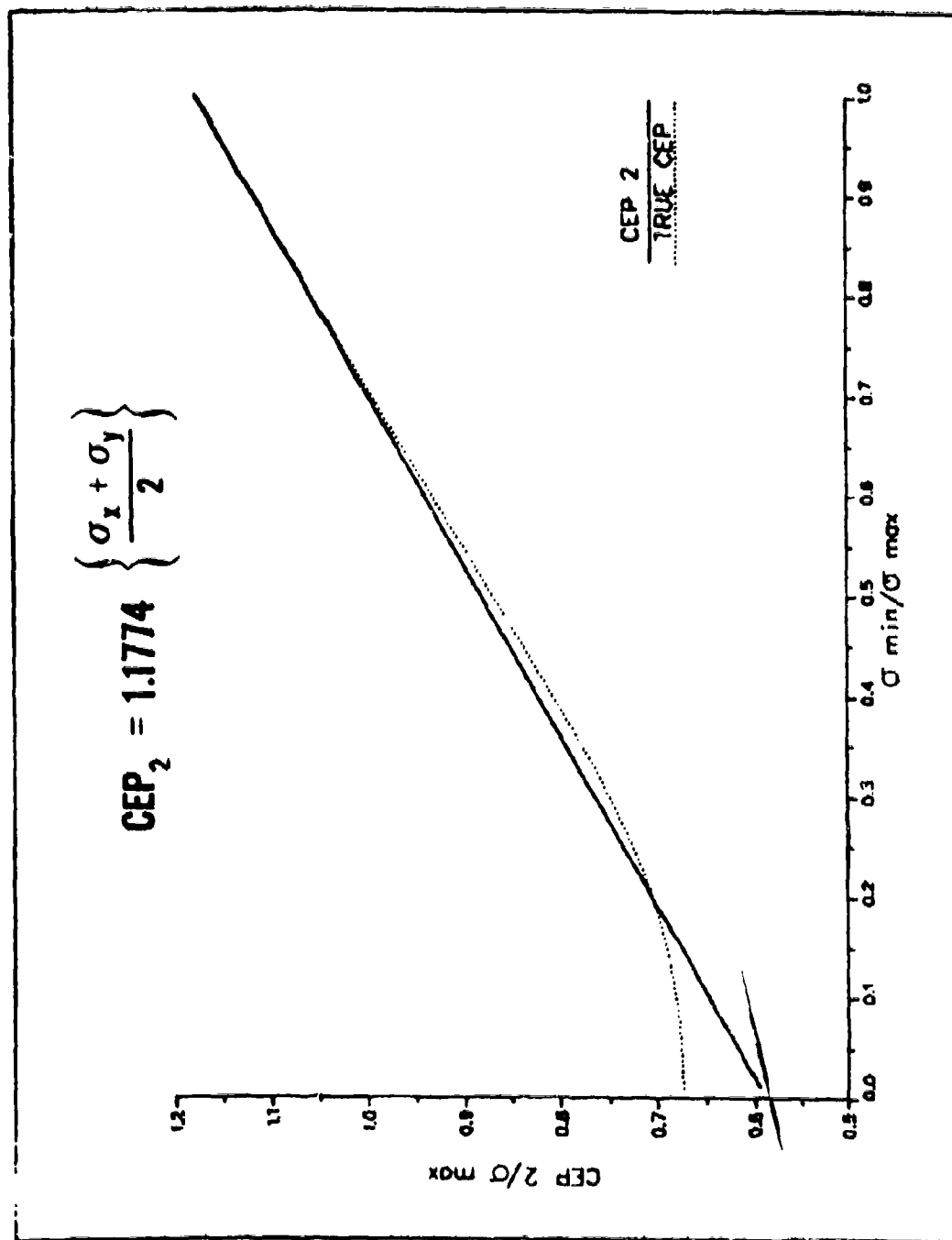


Figure 2

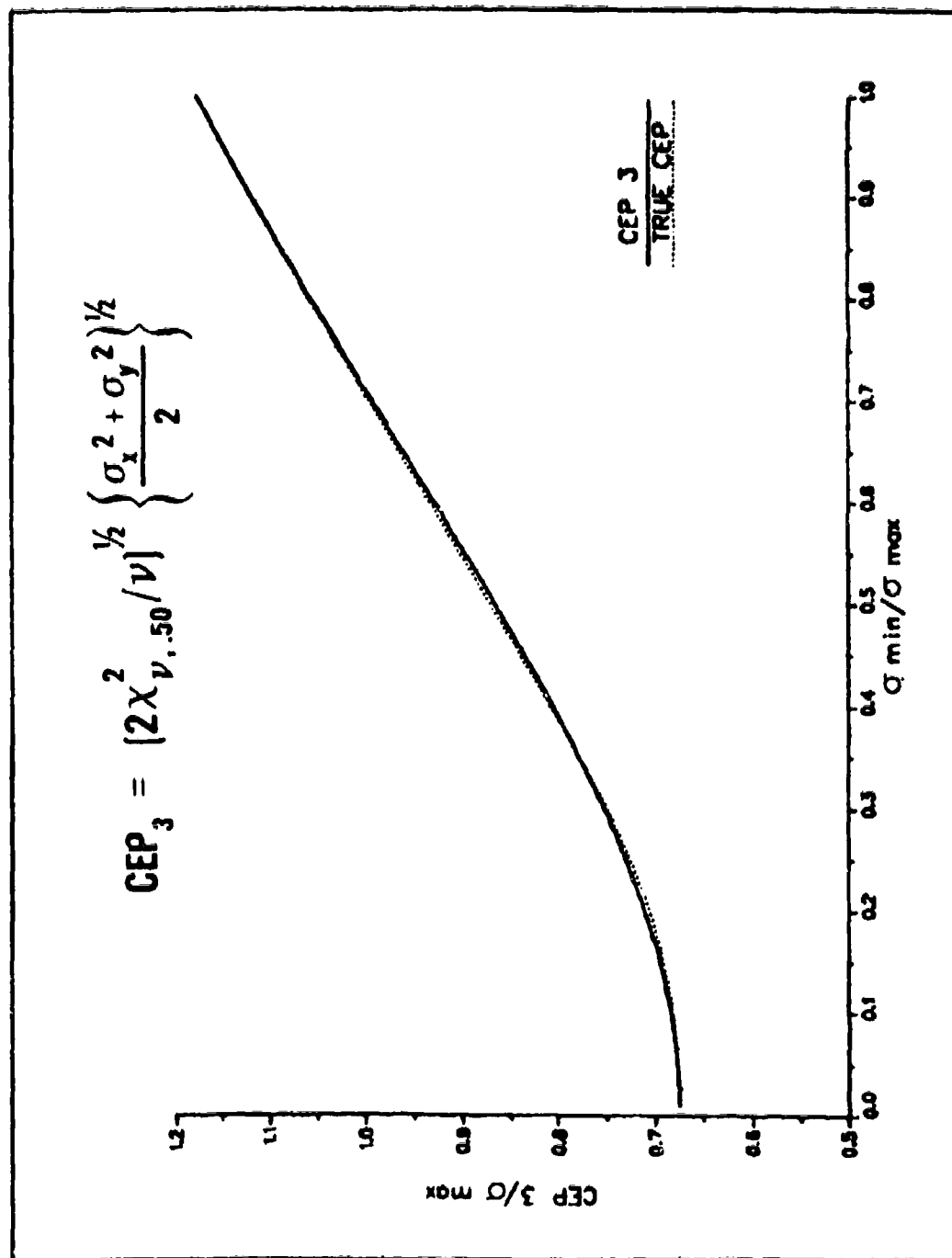


Figure 3

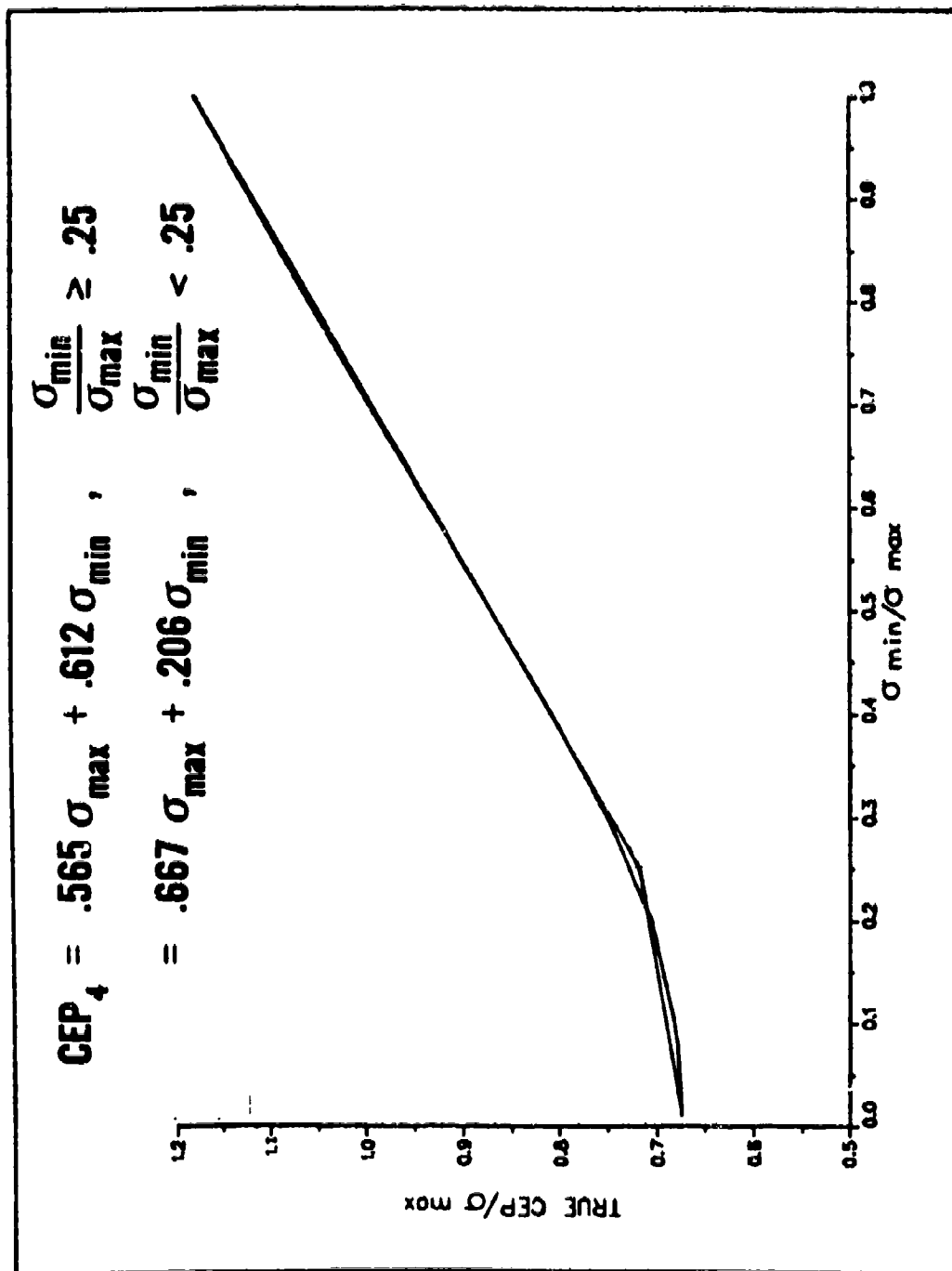


Figure 4

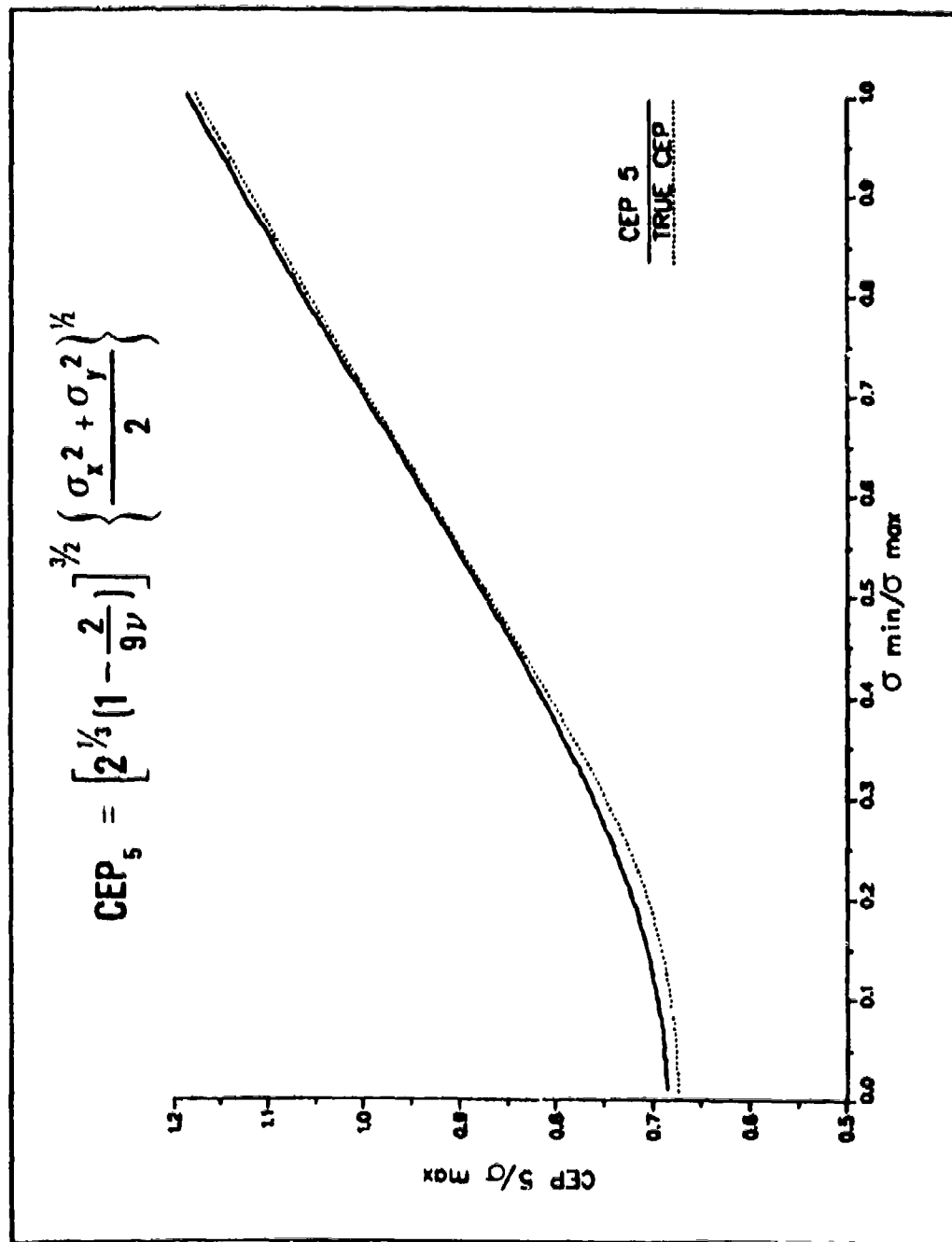


Figure 5

# Approximate Distributions of Estimators

$$(1) \quad \frac{2n \hat{CEP}_1^2}{\sigma_Y^2 K_1^2 \left(\frac{c^2+1}{2}\right)} \sim \chi_{2n}^2$$

$$(2) \quad \frac{\nu^* \hat{CEP}_2^2}{\sigma_Y^2 K_1^2 \left(\frac{c+1}{2}\right)^2} \sim \chi_{\nu^*}^2$$

$$(3) \quad \frac{\nu' \hat{CEP}_3^2}{\sigma_Y^2 K_3^2 \left(\frac{c^2+1}{2}\right)} \sim \chi_{\nu'}^2$$

$$(4) \quad \frac{\nu^* \hat{CEP}_4^2}{\sigma_Y^2 (g_1 + g_2 c)^2} \sim \chi_{\nu^*}^2$$

$$(5) \quad \frac{\nu' \hat{CEP}_5^2}{\sigma_Y^2 K_5^2 \left(\frac{c^2+1}{2}\right)} \sim \chi_{\nu'}^2$$

$$K_1 = 1.1774$$

$$K_3 = (2 \chi_{\nu, .50/\nu})^{1/2}$$

$$K_5 = \left[ 2^{1/3} \left(1 - \frac{2}{9\nu}\right) \right]^{3/2}$$

$$c = \sigma_x / \sigma_y \leq 1$$

$$\nu = \frac{(c^2+1)^2}{c^4+1}$$

$$\nu' = n \nu$$

$$g_1 = .565, \quad g_2 = .612 \quad \text{when } c \geq .25$$

$$g_1 = .667, \quad g_2 = .206 \quad \text{when } c < .25$$

Figure 6

## MEAN SQUARED ERRORS

$$\text{MSE}(\hat{\text{CÉP}}_1) \approx \sigma_y^2 K_1^2 \left( \frac{c^2 + 1}{2} \right) [1 - H^2(2n)] + [\text{Bias}(\hat{\text{CÉP}}_1)]^2$$

$$\text{MSE}(\hat{\text{CÉP}}_2) = \sigma_y^2 K_1^2 \left( \frac{c^2 + 1}{4} \right) [1 - H^2(n)] + [\text{Bias}(\hat{\text{CÉP}}_2)]^2$$

$$\text{MSE}(\hat{\text{CÉP}}_3) \approx \sigma_y^2 K_3^2 \left( \frac{c^2 + 1}{2} \right) [1 - H^2(\nu')] + [\text{Bias}(\hat{\text{CÉP}}_3)]^2$$

$$\nu' = n\nu \qquad \nu = \frac{(c^2 + 1)^2}{c^4 + 1}$$

$$\text{MSE}(\hat{\text{CÉP}}_4) = \sigma_y^2 (g_1^2 + g_2^2 c^2) [1 - H^2(n)] + [\text{Bias}(\hat{\text{CÉP}}_4)]^2$$

$$\text{MSE}(\hat{\text{CÉP}}_5) \approx \sigma_y^2 K_5^2 \left( \frac{c^2 + 1}{2} \right) [1 - H^2(\nu')] + [\text{Bias}(\hat{\text{CÉP}}_5)]^2$$

Exact

Figure 7

# AVERAGE SQUARED ERROR

N = 5

<u>C</u>	<u>ASE 1</u>	<u>ASE 2</u>	<u>ASE 3</u>	<u>ASE 4</u>	<u>ASE 5</u>	<u>S.E.</u>
1.0	.068	.069	.070	.069	.069	.003
.75	.056	.053	.053	.053	.053	.002
.50	.056	.042	.041	.041	.042	.002
.35	.063	.037	.038	.038	.039	.002
.20	.073	.035	.039	.040	.041	.002
.05	.079	.041	.044	.043	.045	.002

N = 10

<u>C</u>	<u>ASE 1</u>	<u>ASE 2</u>	<u>ASE 3</u>	<u>ASE 4</u>	<u>ASE 5</u>	<u>S.E.</u>
1.0	.034	.035	.035	.035	.035	.002
.75	.029	.027	.027	.027	.027	.002
.50	.031	.021	.021	.021	.021	.001
.35	.038	.019	.019	.019	.020	.001
.20	.048	.018	.020	.021	.021	.001
.05	.053	.022	.022	.022	.023	.001

N = 20

<u>C</u>	<u>ASE 1</u>	<u>ASE 2</u>	<u>ASE 3</u>	<u>ASE 4</u>	<u>ASE 5</u>	<u>S.E.</u>
1.0	.017	.017	.017	.017	.017	.001
.75	.014	.013	.013	.013	.013	.001
.50	.017	.011	.010	.010	.010	.001
.35	.024	.010	.009	.009	.010	.001
.20	.034	.009	.010	.011	.011	.001
.05	.039	.013	.011	.011	.012	.001

Figure 8  
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# MEAN SQUARED ERROR

N = 50

<u>C</u>	<u>MSE 1</u>	<u>MSE 2</u>	<u>MSE 3</u>	<u>MSE 4</u>	<u>MSE 5</u>
1.0	.0069	.0069	.0069	.0069	.0070
.75	.0055	.0054	.0057	.0054	.0057
.50	.0077	.0044	.0051	.0042	.0052
.35	.0143	.0041	.0049	.0037	.0050
.20	.0235	.0036	.0047	.0036	.0050
.05	.0275	.0072	.0045	.0042	.0048

N = 100

<u>C</u>	<u>MSE 1</u>	<u>MSE 2</u>	<u>MSE 3</u>	<u>MSE 4</u>	<u>MSE 5</u>
1.0	.0035	.0035	.0035	.0035	.0035
.75	.0029	.0027	.0028	.0027	.0029
.50	.0057	.0023	.0026	.0021	.0026
.35	.0126	.0022	.0024	.0019	.0026
.20	.0220	.0018	.0023	.0018	.0026
.05	.0261	.0053	.0023	.0021	.0025

N = 400

<u>C</u>	<u>MSE 1</u>	<u>MSE 2</u>	<u>MSE 3</u>	<u>MSE 4</u>	<u>MSE 5</u>
1.0	.0009	.0009	.0009	.0009	.0009
.75	.0009	.0007	.0007	.0007	.0007
.50	.0042	.0007	.0007	.0005	.0007
.35	.0113	.0008	.0006	.0005	.0007
.20	.0209	.0005	.0006	.0005	.0008
.05	.0251	.0039	.0006	.0005	.0007

Figure 9

# CONFIDENCE INTERVAL LENGTHS

$N = 5$

<u>C</u>	<u>CL 1</u>	<u>CL 2</u>	<u>CL 3</u>	<u>CL 4</u>	<u>CL 5</u>
1.0	1.213	1.160	1.305	1.145	1.317
.75	1.071	1.028	1.175	1.014	1.186
.50	.950	.923	1.118	.918	1.131
.35	.897	.889	1.129	.917	1.144
.20	.856	.876	1.149	.985	1.168
.05	.841	.912	1.176	1.054	1.196

$N = 10$

<u>C</u>	<u>CL 1</u>	<u>CL 2</u>	<u>CL 3</u>	<u>CL 4</u>	<u>CL 5</u>
1.0	.792	.776	.817	.768	.823
.75	.698	.685	.735	.676	.741
.50	.622	.614	.697	.601	.705
.35	.587	.586	.693	.586	.702
.20	.564	.573	.694	.636	.706
.05	.553	.579	.695	.662	.707

$N = 20$

<u>C</u>	<u>CL 1</u>	<u>CL 2</u>	<u>CL 3</u>	<u>CL 4</u>	<u>CL 5</u>
1.0	.537	.533	.545	.531	.549
.75	.475	.472	.492	.467	.496
.50	.423	.421	.465	.411	.470
.35	.400	.401	.460	.393	.466
.20	.384	.388	.455	.429	.462
.05	.378	.387	.453	.441	.461

Figure 10

# SIMULATED CONFIDENCE LEVELS

**N = 5**

<u>C</u>	<u>PROB 1</u>	<u>PROB 2</u>	<u>PROB 3</u>	<u>PROB 4</u>	<u>PROB 5</u>
1.0	.950	.947	.963	.946	.963
.75	.941	.947	.965	.945	.965
.50	.894	.941	.968	.944	.967
.35	.830	.937	.963	.939	.961
.20	.753	.932	.952	.923	.950
.05	.714	.930	.950	.936	.948

**N = 10**

<u>C</u>	<u>PROB 1</u>	<u>PROB 2</u>	<u>PROB 3</u>	<u>PROB 4</u>	<u>PROB 5</u>
1.0	.947	.945	.955	.944	.955
.75	.935	.944	.958	.943	.959
.50	.876	.941	.967	.943	.966
.35	.789	.939	.967	.941	.965
.20	.689	.939	.959	.931	.955
.05	.640	.931	.951	.943	.948

**N = 20**

<u>C</u>	<u>PROB 1</u>	<u>PROB 2</u>	<u>PROB 3</u>	<u>PROB 4</u>	<u>PROB 5</u>
1.0	.952	.952	.956	.951	.956
.75	.938	.951	.960	.951	.960
.50	.858	.945	.970	.948	.969
.35	.724	.942	.970	.947	.968
.20	.567	.946	.961	.937	.955
.05	.506	.912	.950	.946	.946

Figure 11